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# For Graphs There Are Only Four Types of Hereditary Ramsey Classes

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We characterize all hereditary Ramsey classes of graphs. 

## INTRODUCTION

A Ramsey class is defined as a hereditary class $K$ of structures which has the $A$-partition property for every $A \in K$; see [4, 6]. This notion generalizes the classical Ramsey theorem [8] which in this setting claims that the class of all finite complete graphs is Ramsey. Ramsey classes of graphs, relations, and set systems were studied in [1, 6, 7] where several basic classes were proved to be Ramsey. Here we combine these results with a result of Lachlan and Woodrow [3] and we obtain a complete list of Ramsey classes of graphs. As we shall see there are exactly four basic types of classes of graphs which are Ramsey. All other classes may be obtained by unions.

It should be stressed that being a Ramsey class is a very restrictive property and usually it is not easy to establish this fact. We define below a much weaker property ("pair-Ramsey class"); nevertheless we prove that Ramsey and pair-Ramsey properties coincide. This is quite surprising and there is no direct proof of this fact (our proof is indirect via [3]).

The paper is organized as follows:

Section 1 contains basic notions and properties of Ramsey classes of graphs; Section 2 reviews amalgams and amalgamation classes introduced in [3]; in Section 3 we define the notion of a pair-Ramsey class and derive basic properties. In Section 4 we prove the main result.

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## 1. Ramsey Classes

We deal with undirected graphs $G=(V,E)$ only. If $\leqslant$ is a (total) ordering of $V$ then $(G,\leqslant)$ (and for brevity we write $G$ only) is called an *ordered graph*; $\leqslant$ is also called a *standard ordering of* $G$.

Let $G$, $H$ be graphs. A mapping $F\colon V(G)\to V(H)$ is said to be an *embedding* of $G$ into $H$ iff it is 1–1 and $\{x,y\}\in E(G)$ iff $\{f(x),f(y)\}\in E(H)$. If $(G,\leqslant)$ and $(H,\leqslant)$ are ordered graphs then an (ordered) *embedding* $f$ of $(G,\leqslant)$ into $(H,\leqslant)$ is an embedding of $G$ into $H$ which is, moreover, a monotone mapping with respect to the standard orderings. If $V(G)\subseteq V(H)$ and the inclusion is an embedding then $G$ is said to be a *subgraph* of $H$ (thus all our subgraphs are "induced"). We say that a class $K$ is *hereditary* if $G\in K$ providing that $G$ is isomorphic to a subgraph of some $H\in K$. All classes considered in this paper are hereditary.

Denote by $\binom{H}{G}$ the set of all subgraphs of $H$ which are isomorphic to $G$. If $H$ and $G$ are ordered and we are interested in ordered embeddings, we write $\binom{H}{G}_{\leqslant}$. (An isomorphism is of course an embedding onto. Note that between two ordered graphs there exists at most one isomorphism.)

Let $K$ be a class of graphs, $A\in K$. We say that $K$ has the *$A$-partition property* if for every graph $B$ and for any standard orderings of $A$ and $B$ there exists an ordered graph $C$ with the following property:

For every partition $\binom{C}{A}_{\leqslant}=a_1\cup a_2$ there exists an ordered graph $B'\in\binom{C}{B}_{\leqslant}$ such that $\binom{B'}{A}_{\leqslant}\subseteq a_i$ for either $i=1$ or $i=2$. $(*)$

The validity of statement $(*)$ will be denoted by $C\to(B)^A_2$.

*Note.* It is irrelevant whether we consider partitions into two or finitely many classes. On the other hand we have to deal with ordered graphs; see [6] for a full discussion of these facts.

If a hereditary class $K$ has the $A$-partition property for every $A\in K$ then $K$ is said to be a *Ramsey class*.

*Examples of Ramsey Classes*

1. Compl — The class of all finite complete graphs [8].

2. Gra$(k)$ — The class of all graphs which do not contain $K_{k+1}$—the complete graph with $k+1$ vertices [6, 7].

3. Eq — The class of all equivalences, i.e., graphs which are disjoint unions of complete graphs. (This is well known and it follows by a standard product argument.)

Further examples may be obtained by means of two constructions: Let $K$ be a class of graphs. Denote by $\bar{K}$ the class of all complements $\bar{A}$ of graphs $A\in K$. It is clear that $\bar{K}$ is Ramsey iff $K$ is Ramsey. Note that

Compl $=$ Gra(1). Note that $\overline{\mathrm{Eq}}$ is the class of all complete multipartite graphs (or Turán graphs).

$K \cup K'$ denotes the class of graphs which belong either to $K$ or $K'$. It follows directly from the definition that the union of Ramsey classes is again a Ramsey class. Thus, for example, the class of all graphs $\mathrm{Gra} = \bigcup_{k \geqslant 1} \mathrm{Gra}(k)$ is a Ramsey class. Classes Gra($k$), $\overline{\mathrm{Gra}(k)}$, Eq and $\overline{\mathrm{Eq}}$, and all their unions are called *special.*

## 2. Amalgam

Let $F, G_1, G_2$, be graphs, let $f_i: F \to G_i$, $i = 1, 2$, be embeddings. An *amalgam* of graphs $G_1$ and $G_2$ with respect to embeddings $f_1$ and $f_2$ is any graph $H$ for which there are embeddings $g_i: G_i \to G$, $i = 1, 2$, such that $g_1 \circ f_1 = g_2 \circ f_2$.

*Note.* An amalgam is not uniquely determined, and this definition is different from the (more usual) category theory definition of an amalgam (as a pushout of mappings $f_1$ and $f_2$) which is up to an isomorphism uniquely determined (compare [5]).

Let $K$ be a class of graphs. We say that $K$ *has amalgams* if for every choice of graphs $F$, $G_1$, $G_2$, and embeddings $f_i: F \to G_i$, $i = 1, 2$, there exists an amalgam in $K$. Note that the classes Eq, $\overline{\mathrm{Eq}}$, Gra($k$) and $\overline{\mathrm{Gra}(k)}$ have amalgams. We say that a class $K$ has *disjoint copies* if for every $A \in K$ there exists $B \in K$ and graphs $A_1, A_2 \in \binom{B}{A}$ which are vertex disjoint. We shall make use of the following consequence of a deep result of Lachlan and Woodrow.

Theorem 2. *The following classes and their unions are the only hereditary classes with disjoint copies which have amalgams:*

$$\mathrm{Gra}(k), \quad \overline{\mathrm{Gra}(k)}, \quad \mathrm{Eq} \cap \mathrm{Gra}(k),$$
$$\overline{\mathrm{Eq}} \cap \mathrm{Gra}(k), \quad \overline{\mathrm{Eq}} \cap \overline{\mathrm{Gra}(k)}, \quad \mathrm{Eq} \cap \overline{\mathrm{Gra}(k)}.$$

## 3. Pair-Ramsey Classes

We say that the hereditary class $K$ of graphs is *pair-Ramsey* if for every $A, B \in K$ and for every two vertex disjoint graphs $A_1, A_2 \in \binom{B}{A}$ there exist $C \in K$ such that

for every coloring $\binom{C}{A} = a_1 \cup a_2$ there exists an embedding $g: B \to C$ such that $g(A_1) \in a_i$ iff $g(A_2) \in a_i$, $i = 1, 2$ ($g(A_i)$ is the image of $A_i$ in $B$). (**)

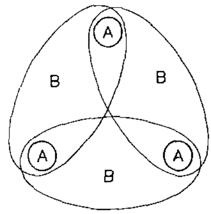


FIGURE 1

We write $C \Rightarrow B$ (with respect to $A$) if the graph $C$ satisfies this property for all choices of $A_1$, $A_2$ in $\binom{B}{A}$. Explicitly, $C \Rightarrow B$ if for every $A$ and for every choice of vertex disjoint $A_1, A_2 \in \binom{B}{A}$ and for every coloring $\binom{C}{A} = a_1 \cup a_2$ there exists an embedding $g\colon B \to C$ ($g$ depends on the choice of $A_1$, $A_2$, and the partition) such that $g(A_1) \in a_i$ iff $g(A_2) \in a_i$.

Let us remark that given $A_1, A_2$, and $B$ it is very easy to construct an example of a graph $C$. This is schematically depicted of Fig. 1.

This is in sharp contrast with difficulties which are related to a construction of graph $C$ which satisfics $(*)$ (i.e., $C \to (B)_2^A$).

Therefore it is a bit surprising that the classes of graphs which satisfy $(*)$ and $(**)$ for every choice of graphs $A$ and $B$ are identical.

## 4. MAIN RESULT

We prove here:

THEOREM. *For a hereditary class of graphs K the following statements are equivalent*:

1. *K is Ramsey*;
2. *K is pair-Ramsey with disjoint copies*;
3. *K is special.*

*Proof.* We prove $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$1 \Rightarrow 2$. Fix vertex disjoint $A_1, A_2 \in \binom{B}{A}$ and fix an order $\leqslant$ of $B$ such that $A_1$ and $A_2$ with inherited order $\leqslant$ are monotone isomorphic to the ordered graph $(A, \leqslant)$. Let $C$ be an ordered graph with $C \to (B)_2^A$ (i.e., with $(*)$). But this in turn implies (much weaker) $(**)$.

We still have to prove that $K$ has disjoint copies. However, this is easy as for given $B$ every $C \to (B)_3^{K_1}$ contains vertex disjoint copies $B_1, B_2 \in \binom{C}{B}$. (Here $K_1$ is the singleton graph and $C \to (B)_3^{K_1}$ denotes the validity of $(*)$ for partitions with 3 classes $a_1, a_2, a_3$.)

$2 \Rightarrow 3$. The main step consists in proving the following:

*Claim.* Let $K$ be a pair-Ramsey hereditary class with disjoint copies. Then $K$ is a union of amalgamation classes having disjoint copies.

*Proof.* Fix $G \in K$. Define recursively graphs $G_1, G_2, \ldots,$ from $K$ as follows:

$$G_1 = G$$

$$G_{i+1} \Rightarrow G_i \text{ and } G_{i+1} \text{ contains two disjoint copies of } G_i,\ i \geq 1.$$

(see Section 3 for definition of $\Rightarrow$).

Let $K_G$ be the class of all subgraphs of graphs $G_i$, $i = 1, 2, \ldots$ . We prove that $K_G$ is an amalgamation class. Towards this end let $A$, $B_1$, $B_2$ be graphs belonging to $K_G$, let $f_i$: $A \to B_i$ be given embeddings.

Clearly we may suppose without loss of generality that $B = B_1 = B_2 = G_r$ for some $r \geq 1$. By our choice $G_{r+1} = G_s$ contains two vertex disjoint copies $B^1, B^2 \in \binom{G_s}{B}$. Let $\Phi_i$: $B \to B^i$, $i = 1, 2$, be an isomorphism. Put $A^i = \Phi_i \circ f_i(A) \in \binom{G_s}{A}$.

Finally let $C \in K_G$ be a graph which satisfies (**) with respect to $G_s$ and let $A^1$, $A^2$, (e.g., $C = G_{s+1}$). But then there are subgraphs $G^1, G^2 \in \binom{C}{G_s}$ and isomorphisms $g_i$: $G_s \to G^i$, $i = 1, 2$, such that $g_1 \circ \Phi_1 \circ f_1(A) = g_2 \circ \Phi_2 \circ f_1(A)$ (for otherwise we could color by color 1 the subgraphs of $C$ which correspond to $A^1$ in a copy of $G_s$ and the other copies of $A$ we could color by 2; this particular coloring violates (**)).

But then the subgraph of $C$ induced by the set $V(G^1) \cup V(G^2)$ induces an amalgamation of $B_1$, $B_2$ with respect to embeddings $f_1$: $A \to B_1$ and $f_2$: $A \to B_2$.

This proves the Claim.

Now 3 follows by applying the Lachlan–Woodrow theorem: we know that $K$ is a union of (some) classes of types $\mathrm{Gra}(k)$, $\mathrm{Eq} \cap \mathrm{Gra}(k)$, $\overline{\mathrm{Eq}} \cap \mathrm{Gra}(k)$, and their complements. We shall prove that $K$ is the union of some classes $\mathrm{Gra}(k)$, $\overline{\mathrm{Gra}(k)}$, Eq and $\overline{\mathrm{Eq}}$. For contradiction suppose, e.g., $K \supseteq \mathrm{Eq} \cap \mathrm{Gra}(k)$ for some $k > 1$ and let $K \not\supseteq \mathrm{Gra}(k)$, $K \not\supseteq \mathrm{Eq}$. Take maximal such $k$. Assume also that $K \not\supseteq \overline{\mathrm{Gra}(n)}$ for some $n$ (as otherwise $K = \mathrm{Gra}$). Now let $B$ be the disjoint union of $N > n$ complete graphs of size $k$. It is $B \in K$ and if $C \Rightarrow B$, $C \in K$, then $C \notin \overline{\mathrm{Eq}}$ and thus $C \in \mathrm{Eq}$. As $N > n$ was arbitrary we have easily $K \supseteq \mathrm{Eq} \cap \mathrm{Gra}(2k-1)$, a contradiction. We proceed analogously for other classes $\mathrm{Eq} \cap \overline{\mathrm{Gra}(k)}$ and complements.

$3 \Rightarrow 1$. The special classes of graphs were proved to be Ramsey in [6]; see part 1 of this paper. ∎

Let us remark that the characterisation of Ramsey classes of set systems (even for 3-uniform hypergraphs) is not known despite the fact that [6, 7] contains a large variety of Ramsey classes. In fact, we conjecture that there are no other hereditary Ramsey classes.

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